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Geometrical aspects of operator ordering terms in gauge invariant quantum models

P J Houston

School of Theoretical Physics, Dublin Institute for Advanced Studies, 10 Burlington Road, Dublin 4, Ireland

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Abstract. Finite-dimensional quantum models with both boson and fermion degrees of freedom, and which have a gauge invariance, are studied here as simple versions of gauge invariant quantum field theories. The configuration space of these finite-dimensional models has the structure of a principal fibre bundle and has defined on it a metric which is invariant under the action of the bundle or gauge group. When the gauge-dependent degrees of freedom are removed, thereby defining the quantum models on the base of the principal fibre bundle, extra operator ordering terms arise. By making use of dimensional reduction methods in removing the gauge dependence, expressions are obtained here for the operator ordering terms which show clearly their dependence on the geometry of the principal fibre bundle structure.

1. Introduction

The presence of gauge invariance in any Lagrangian means that the Lagrangian is singular, i.e. it has redundant degrees of freedom (see [1] and references therein). In order to proceed with the quantisation of the Lagrangian these redundant degrees of freedom must be removed. In gauge invariant field theories, for example Yang-Mills field theory, when the gauge dependence is removed from the configuration space the resulting space of gauge orbits is known to be equipped with a curved Riemannian geometry [2-6]. Hence, in order to consider quantum mechanics on such a curved configuration space (even though infinite dimensional in the case of a field theory) a choice has to be made for the ordering of operators in the Hamiltonian operator. This operator ordering problem for Yang-Mills field theory was considered by Christ and Lee [7] and Gawedzki [8] where extra operator ordering terms are found. However, their approach does not fully bring out and identify the underlying geometric features that are present due to the gauge invariance. The two main geometric structures on the configuration space of, for example, Yang-Mills field theory are first that of a principal fibre bundle and second that of a Riemannian manifold where the metric is invariant under the action of the gauge group [3]. It is not necessary, of course, to have infinite dimensionality, i.e. a field theory, in order that the above geometric structures occur. There are examples of finite-dimensional configuration spaces which have these structures. In this paper, in order to concentrate on the geometric point of view, we consider the quantum mechanics of bosons and fermions on finite-dimensional configuration spaces where there is a gauge invariance, i.e. we have a principal fibre

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bundle and a gauge invariant Riemannian metric. We will identify the operator ordering terms with reference to features associated with these geometric structures.

As our motivation originates from gauge invariant field theory, we now briefly consider the example of a single SU(2) Yang–Mills field and identify there the structures we are interested in. The configuration space is the infinite-dimensional space of gauge potentials $A = A(x)$, where each component is valued in $\mathbb{C}^{2\times 2}$ and satisfies $A_i^\dagger = A_i$ and $\mathrm{tr}(A_i) = 0$. The classical dynamical Lagrangian, in the $A^0 = 0$ gauge, is given by

$$L(t) = \frac{1}{2}\int \mathrm{d}_3 x \left\{ \mathrm{tr}\left(\frac{\partial A}{\partial t} \cdot \frac{\partial A}{\partial t}\right) - V(A)\right\} \tag{1.1}$$

together with the Gauss law

$$\sum_i \left\{ \frac{\partial}{\partial x}{}_i\left(\frac{\partial A_i}{\partial t}\right) + \left[A_i, \frac{\partial A_i}{\partial t}\right]\right\} = 0. \tag{1.2}$$

Here, the potential $V(A)$ is the space integral of the trace of the magnetic field squared. The infinite-dimensional gauge group is $\{U = U(x)\}$, where $U(x)$ is a unitary unimodular element of $\mathbb{C}^{2\times 2}$. This gauge group acts on the configuration space as follows:

$$A_i(x) \to A_i^U(x) = U^{-1}(x) A_i(x) U(x) + U^{-1}(x) \frac{\partial U}{\partial x^i}(x) \tag{1.3}$$

and leaves equations (1.1) and (1.2) invariant. The configuration space together with the group action of (1.3) essentially gives the structure of a principal fibre bundle†. We have also a metric defined implicitly on the configuration space, namely the Euclidean metric, due to the form of the kinetic energy term in $L(t)$. This infinite-dimensional metric is invariant under the gauge group action. Moreover, as can be seen directly, the transformation (1.3) is contained within the Euclidean group action on the configuration space. The first term of A^U is a rotation about the origin of the configuration space and the second term of A^U is a translation.

In considering finite-dimensional models to illustrate the geometric situation pertaining to gauge field theories, we take two points of view. (Of course, by restricting to a finite-dimensional configuration space, we are putting aside questions of renormalisation and Lorentz covariance which are present in gauge field theory.) For the first case (case A) we require that the finite-dimensional configuration space be equipped with the Euclidean metric and that the gauge group action on the configuration space lie non-trivially within both the rotation and translation parts of the Euclidean group action. Because of finite dimensionality, examples fulfilling the conditions for case A are somewhat limited. Indeed, since the translation group is Abelian, it is clear that the only gauge groups that will be of interest will also be Abelian. Thus, for a typical example covering case A we take the configuration space to be three-dimensional Euclidean space and the gauge group to be $U(1)$. We consider this example in section 2. Case A is the direct finite-dimensional analogy of the situation in Yang–Mills field theory, but too few examples are possible to allow general conclusions to be drawn. Hence, for the second case (case B) we adopt a more general stance. We take the configuration space to be a finite-dimensional Riemannian manifold with a metric which is invariant under the free action of a finite-dimensional Lie group G. We consider this case in section 3.

† For a principal fibre bundle the group is required to act freely. For (1.3) to give a free group action some restrictions are needed on $\{A(x)\}$ (see [3]).

To consider the quantum mechanics of any gauge invariant or singular Lagrangian we must use Dirac's method of constraints (see [1]). In this method we are given a closed operator algebra consisting of the Hamiltonian operator and the constraint operators. The quantum mechanics is then defined by the Hamiltonian operator but as operating on wavefunctions which are annihilated by the constraint operators. Proceeding further, by transforming the Hamiltonain so that it operates on wavefunctions of the space of gauge orbits we can read off the extra operator ordering terms. The ordering of operators of the transformed final Hamiltonian will depend, of course, on the choice of the ordering of operators in the original Hamiltonian, i.e. prior to the application of the constraint operators. For case A the original configuration space is flat Euclidean space. Thus we make the canonical choice of $-\frac{1}{2}$ times the Euclidean Laplacian for the kinetic energy term of the original Hamiltonian. For case B the original configuration space is already curved. Here, we make the natural geometric operator ordering choice that the kinetic energy term of the original Hamiltonian is $-\frac{1}{2}$ times the Laplacian on the original configuration space.

We study these operator ordering terms with reference to the geometric features associated with a principal fibre bundle on which is defined an invariant Riemannian metric. These features are the connection form and the Riemannian structures of the base and the fibres of the principal fibre bundle. The geometric situation that we have here has also occurred in the context of the dimensional reduction of Kaluza-Klein-type theories (see [9, 10] and the review [11]). Thus in the geometric discussion we have found it convenient to take our notation and approach from there.

Finally, section 4 contains the concluding remarks.

2. A three-dimensional model with a one-dimensional gauge invariance

In this section we consider a three-dimensional example typifying most directly the geometrical situation in gauge invariant field theories, i.e. the case A referred to in the introduction. We will consider first the physics point of view in approaching a boson-only model and a boson-plus-fermion model. Then we will adopt a geometric point of view in looking at these models.

The Lagrangian for a boson particle moving in three-dimensional Euclidean space, which is periodic with period $2\pi\beta$ in the z direction, we take to be

$$L'_B = \tfrac{1}{2}\{(\dot{x}+qy)^2+(\dot{y}-qx)^2+(\dot{z}-\beta q)^2\} - V(r, \theta - z/\beta). \tag{2.1}$$

Here, r and θ are the polar variables to x and y, and V is an arbitrary function of r and $\theta - z/\beta$; the variable q plays the same role here as A^0 does in Yang-Mills field theory and the dot denotes time derivative. The Lagrangian is invariant under the gauge transformation

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \to \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \beta\alpha \end{pmatrix} \tag{2.2a}$$

$$q \to q + \dot{\alpha}. \tag{2.2b}$$

In the $A^0 = 0$ (i.e. $q = 0$) gauge the Lagrangian becomes

$$L_B = \tfrac{1}{2}\dot{x}^2 - V(r, \theta - z/\beta) \tag{2.3}$$

but in addition we have the Gauss-law constraint arising from the equation of motion for the q variable

$$x\dot{y} - y\dot{x} + \beta\dot{z} = 0. \tag{2.4}$$

The similarity of this three-dimensional model with Yang-Mills field theory can be seen by comparing the Lagrangians of equations (1.1) and (2.3) and the Gauss laws (1.2) and (2.4). Furthermore, the three-dimensional configuration space has the Euclidean metric defined on it due to the form of the kinetic energy term in equation (2.3) and the remaining time-independent gauge transformation (2.2a), which leaves the $A^0 = 0$ gauge Lagrangian (2.3) invariant, lies in both the rotation and translation parts of the Euclidean group.

The momentum conjugate to $\boldsymbol{x}$ is $\boldsymbol{p} = \dot{\boldsymbol{x}}$ using equation (2.3). Hence, the Hamiltonian is given by

$$H_B = \tfrac{1}{2}\boldsymbol{p}^2 + V(r, \theta - z/\beta). \tag{2.5}$$

This is subject to the Gauss-law constraint (2.4) which in terms of the momenta becomes

$$xp_y - yp_x + \beta p_z = 0. \tag{2.6}$$

In order to quantise the constrained system (2.5) and (2.6) we let $\boldsymbol{p}$ become $(1/\mathrm{i})\boldsymbol{\nabla}$† and make the canonical operator ordering choice for the Hamiltonian of a particle moving in Euclidean space, i.e.

$$\hat{H}_{\mathrm{B}} = -\tfrac{1}{2}\boldsymbol{\nabla}^2 + V(r, \theta - z/\beta). \tag{2.7}$$

According to Dirac's method [1] the constraint that is imposed on the wavefunction is

$$(x\partial/\partial y - y\partial/\partial x + \beta\partial/\partial z)\psi = 0. \tag{2.8}$$

The quantum mechanics of the boson model is defined by the Hamiltonian operator (2.7) acting on wavefunctions which satisfy equation (2.8).

In using the coordinates (x, y, z), the similarity of our model with Yang-Mills field theory is clearly displayed. However, it will be more convenient for future calculations to change to cylindrical coordinates (r, θ, z). Thus in terms of these coordinates the original Lagrangian becomes

$$L'_{\mathrm{B}} = \tfrac{1}{2}\{\dot{r}^2 + r^2(\dot{\theta} - q)^2 + (\dot{z} - \beta q)^2\} - V(r, \theta - z/\beta) \tag{2.1$'$}$$

and the gauge transformation (2.2a) becomes

$$\begin{aligned} &r \to r \\ &\theta \to \theta + \alpha \\ &z \to z + \beta\alpha. \end{aligned} \tag{2.2a'}$$

In the $A^0 = 0$ gauge the Lagrangian and Gauss law are as follows:

$$L_{\mathrm{B}} = \tfrac{1}{2}(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2) - V(r, \theta - z/\beta) \tag{2.3$'$}$$

$$r\dot{\theta} + \beta\dot{z} = 0. \tag{2.4$'$}$$

† Because the z variable has the topological structure of a circle, we have a one-parameter family of inequivalent choices for p_z, namely $(1/\mathrm{i})\partial/\partial z + \delta$ with $0 \leq \delta < 1/\beta$. The choice $\delta = 0$ is taken for convenience as the end result is not affected by it.

The momenta conjugate to (r, θ, z) are $(p_r = \dot{r}, p_\theta = r^2\dot{\theta}, p_z = \dot{z})$. The Hamiltonian and Gauss law are given by

$$H_B = \tfrac{1}{2}(p_r^2 + p_\theta^2/r^2 + p_z^2) + V(r, \theta - z/\beta) \tag{2.5'}$$

$$p_\theta + \beta p_z = 0. \tag{2.6'}$$

In quantising, (p_r, p_θ, p_z) become $(1/\mathrm{i})(\partial/\partial r, \partial/\partial\theta, \partial/\partial z)$. In the Hamiltonian operator (2.7) we have only to write the Laplacian ∇^2 in cylindrical coordinates. However, we give $\hat{H}_B$ in the alternative form

$$(\psi_1, \hat{H}_B\psi_2) = \int_0^\infty r\,\mathrm{d}r \int_0^{2\pi} \mathrm{d}\theta \int_0^{2\pi\beta} \mathrm{d}z \left[\frac{1}{2}\left(\frac{\partial\psi_1^*}{\partial r}\frac{\partial\psi_2}{\partial r} + \frac{1}{r^2}\frac{\partial\psi_1^*}{\partial\theta}\frac{\partial\psi_2}{\partial\theta} + \frac{\partial\psi_1^*}{\partial z}\frac{\partial\psi_2}{\partial z}\right) + V(r, \theta - z/\beta)\psi_1^*\psi_2\right] \tag{2.7'}$$

where the Gauss-law constraint on the wavefunctions now becomes

$$\frac{\partial\psi}{\partial\theta} + \beta\frac{\partial\psi}{\partial z} = 0. \tag{2.8'}$$

Equation (2.7′) will give a more suitable formula for $\hat{H}_B$ than (2.7) for later use.

We now consider a modification of the Lagrangian L_B' to produce a gauge invariant model of bosons and fermions which represents a three-dimensional version of a gauge field theory with fermions. For the modified Lagrangian we write (see [12])

$$L' = L_B' + \boldsymbol{\xi}^\dagger[\mathrm{i}\,\mathrm{d}/\mathrm{d}t - \tfrac{1}{2}\mu(x\boldsymbol{\sigma}_1 + y\boldsymbol{\sigma}_2) - \tfrac{1}{2}q\boldsymbol{\sigma}_3]\boldsymbol{\xi} \tag{2.9}$$

where μ is a constant

$$\boldsymbol{\xi} = \begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix} \qquad \boldsymbol{\sigma}_1 = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$$
$$\boldsymbol{\sigma}_2 = \begin{pmatrix}0 & -\mathrm{i}\\ \mathrm{i} & 0\end{pmatrix} \qquad \boldsymbol{\sigma}_3 = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \tag{2.10}$$

and where † denotes the complex conjugate transpose. Here ξ_i are the fermion variables. The similarity in form of the second term of equation (2.9) to that of the gauge potential-fermion field coupling term of a field theory is self-evident. The Lagrangian L' is invariant under the gauge transformation (2.2) together with

$$\boldsymbol{\xi} \to \begin{pmatrix}\mathrm{e}^{-\mathrm{i}\alpha/2} & 0\\ 0 & \mathrm{e}^{\mathrm{i}\alpha/2}\end{pmatrix}\boldsymbol{\xi}. \tag{2.11}$$

In the $A^0 = 0$ (i.e. $q = 0$) gauge the Lagrangian and Gauss-law constraint are

$$L = L_B + \boldsymbol{\xi}^\dagger[\mathrm{i}\,\mathrm{d}/\mathrm{d}t - \tfrac{1}{2}\mu(x\boldsymbol{\sigma}_1 + y\boldsymbol{\sigma}_2)]\boldsymbol{\xi} \tag{2.12}$$

$$r^2\dot{\theta} + \beta\dot{z} + \tfrac{1}{2}\boldsymbol{\xi}^\dagger\boldsymbol{\sigma}_3\boldsymbol{\xi} = 0. \tag{2.13}$$

The quantised Hamiltonian associated with the Lagrangian of equation (2.12) may be written as

$$\hat{H} = \hat{H}_B + \hat{H}_F \tag{2.14}$$

where the boson Hamiltonian operator $\hat{H}_B$ is already given by equation (2.7) or (2.7′). The fermion part of $\hat{H}$ is

$$\hat{H}_F = \tfrac{1}{2}\mu\hat{\boldsymbol{\xi}}^{\dagger}(x\boldsymbol{\sigma}_1 + y\boldsymbol{\sigma}_2)\hat{\boldsymbol{\xi}} \tag{2.15}$$

where

$$\hat{\boldsymbol{\xi}} = \begin{pmatrix} \hat{\xi}_1 \\ \hat{\xi}_2 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} l_1 + \mathrm{i}l_3 \\ l_2 + \mathrm{i}l_4 \end{pmatrix} \tag{2.16}$$

and $\{l_1, l_2, l_3, l_4\}$ satisfy the four-dimensional Clifford algebra

$$l_i l_j + l_j l_i = \delta_{ij}. \tag{2.17}$$

In order to obtain a matrix representation of this Clifford algebra we need to take l_1, l_2, l_3, l_4 (and $\hat{\xi}_1$, $\hat{\xi}_2$) as elements of $\mathbb{C}^{4\times 4}$. A particular matrix representation is given by

$$\hat{\xi}_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \qquad \hat{\xi}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \tag{2.18}$$

and $\hat{\xi}_1^{\dagger}$, $\hat{\xi}_2^{\dagger}$ are obtained by taking their complex conjugate transpose. With respect to this representation, $\hat{H}_F$ can be written as the 4×4 matrix

$$\hat{H}_F = \frac{\mu}{2}\begin{pmatrix} -x\boldsymbol{\sigma}_1 + y\boldsymbol{\sigma}_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} = -\frac{\mu r}{2}\begin{pmatrix} \begin{matrix} 0 & \mathrm{e}^{\mathrm{i}\theta} \\ \mathrm{e}^{-\mathrm{i}\theta} & 0 \end{matrix} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \tag{2.19}$$

where $\mathbf{0}$ denotes the 2×2 matrix of zeros. It follows, of course, that in this matrix representation $\hat{H}$ operates on wavefunctions ψ which are valued in $\mathbb{C}^4$. The Gauss-law constraint on these wavefunctions is

$$\left(\frac{\partial}{\partial\theta} + \beta\frac{\partial}{\partial z} + \frac{\mathrm{i}}{2}\hat{\boldsymbol{\xi}}^{\dagger}\boldsymbol{\sigma}_3\hat{\boldsymbol{\xi}}\right)\psi = \left(\frac{\partial}{\partial\theta} + \beta\frac{\partial}{\partial z} - \frac{\mathrm{i}}{2}\begin{pmatrix} \boldsymbol{\sigma}_3 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\right)\psi = 0. \tag{2.20}$$

Thus the quantum mechanics of the boson–fermion model is defined by the Hamiltonian $\hat{H}$, given by equations (2.14), (2.7) or (2.7′), and (2.19) as acting on $\mathbb{C}^4$-values wavefunctions satisfying equation (2.20).

Having set up the quantum models, we now consider the underlying geometry of the configuration space of these models (see [13]). Because of periodicity in the z direction, the configuration space P is $\mathbb{R}^2\times S^1$. Let (r, θ, z) be the (cylindrical) coordinates of a point in P, then an element $\mathrm{e}^{\mathrm{i}\alpha}$ of the gauge group U(1) acts freely (i.e. without fixed points) on P as follows:

$$P \ni (r, \theta, z) \xrightarrow{\mathrm{e}^{\mathrm{i}\alpha}} (r, \theta+\alpha, z+\beta\alpha) \in P. \tag{2.21}$$

(Alternatively, if we were to use cartesian coordinates on P the $U(1)$ action would be as given by equation (2.2*a*).) The group action (2.21) defines the structure of a principal fibre bundle on P, the fibres being given by $\{(r, \theta+\alpha, z+\beta\alpha) | 0 \leq \alpha < 2\pi\}$. Identifying all points in P which lie on the same fibre we obtain $M = \mathbb{R}^2$ which is the base of the principal fibre bundle. The bundle projection $P \to M$ is then given by

$$P \ni (r, \theta, z) \to (\rho, \phi) = (r, \theta - z/\beta) \in M \tag{2.22}$$

where (ρ, ϕ) are the polar coordinates of a point in M. The Euclidean metric ds_P^2 is defined on P

$$ds_P^2 = dx^2 + dy^2 + dz^2 = dr^2 + r^2\, d\theta^2 + dz^2 \tag{2.23}$$

and the group action in either form (2.2a) or (2.21) leaves ds_P^2 invariant.

The tangent space to P, $\mathcal{T}_P$, is spanned by $\{\partial/\partial r, \partial/\partial\theta, \partial/\partial z\}$. We define a vector field ε which is tangent to the fibres as follows:

$$\begin{aligned} \varepsilon f(r, \theta, z) &= \left[\frac{df}{d\alpha}(r, \theta+\alpha, z+\beta\alpha)\right]_{\alpha=0} \\ &= \left(\frac{\partial}{\partial\theta} + \beta\frac{\partial}{\partial z}\right) f(r, \theta, z). \end{aligned} \tag{2.24}$$

The vertical subspace of $\mathcal{T}_P$, $\mathcal{V}$, is spanned by $\{\varepsilon = \partial/\partial\theta + \beta\partial/\partial z\}$. Using ds_P^2 we can now define a metric, ds_F^2, on each fibre as follows:

$$ds_P^2(\varepsilon, \varepsilon) = r^2 + \beta^2. \tag{2.25}$$

Therefore the fibre above the point (ρ, ϕ) of M has the metric

$$ds_F^2 = (\rho^2 + \beta^2)\, d\alpha^2 \tag{2.26}$$

where α $(0 \leqslant \alpha < 2\pi)$ represents the coordinates of a point on the fibre. (Hence, the length of the fibre above (ρ, ϕ) is $2\pi(\rho^2+\beta^2)^{1/2}$.)

The metric ds_P^2 also allows us to define a horizontal subspace of $\mathcal{T}_P$ which is orthogonal to $\mathcal{V}$ denoted $\mathcal{H}$. We take $\mathcal{H}$ to be spanned by

$$\left\{\frac{\partial}{\partial r}, \frac{1}{r^2}\frac{\partial}{\partial\theta} - \frac{1}{\beta}\frac{\partial}{\partial z}\right\}.$$

Thus $\mathcal{T}_P = \mathcal{V} \oplus \mathcal{H}$. Furthermore, because ds_P^2 is invariant under the U(1) bundle action, the above construction of $\mathcal{H}$ naturally defines a U(1) connection on the principal fibre bundle. The associated connection 1-form w is determined by

$$w(\varepsilon) = 1 \qquad w(\mathcal{H}) = 0. \tag{2.27}$$

Applying these conditions w is computed to be

$$w = \frac{1}{r^2+\beta^2}(r^2\, d\theta + \beta\, dz). \tag{2.28}$$

In order to express w in a more familiar format we need to consider a local section of the bundle, i.e. a local map $M \to P$. In fact, we consider a family of such sections labelled by the function $\sigma = \sigma(\rho, \phi)$ and given by

$$r = \rho \qquad \theta = \phi + \sigma(\rho, \phi) \qquad z = \beta\sigma(\rho, \phi). \tag{2.29}$$

Using the section (2.29) we have

$$\begin{aligned} &dr = d\rho \qquad d\theta = \frac{\partial\sigma}{\partial\rho}\, d\rho + \left(1 + \frac{\partial\sigma}{\partial\phi}\right) d\phi \\ &dz = \beta\frac{\partial\sigma}{\partial\rho}\, d\rho + \beta\frac{\partial\sigma}{\partial\phi}\, d\phi. \end{aligned} \tag{2.30}$$

Therefore w can now be written as a 1-form on M as

$$w=\mathscr{A}_\rho^{(\sigma)}\,\mathrm{d}\rho+\mathscr{A}_\phi^{(\sigma)}\,\mathrm{d}\phi \tag{2.31}$$

where

$$\mathscr{A}_\rho^{(\sigma)}=\frac{\partial\sigma}{\partial\rho}\qquad \mathscr{A}_\phi^{(\sigma)}=\frac{\rho^2}{\rho^2+\beta^2}+\frac{\partial\sigma}{\partial\phi}. \tag{2.32}$$

We note that the field strength (or bundle curvature) is computed to be

$$\mathscr{F}=\mathrm{d}w=\frac{2\rho\beta^2}{(\rho^2+\beta^2)^2}\,\mathrm{d}\rho\,\mathrm{d}\phi \tag{2.33}$$

and that the total flux (or first Chern class) is

$$\frac{1}{2\pi}\int\int_M \mathscr{F}=1.$$

The tangent space to M, $\mathscr{T}_M$ is spanned by $\{\partial/\partial\rho,\partial/\partial\phi\}$. For any function f which is constant on the fibres of P i.e. $f(\rho,\phi)=f(r,\theta-z/\beta)$ we have that $\partial f/\partial\rho=\partial f/\partial r$, $\partial f/\partial\phi=\partial f/\partial\theta$. Thus we can lift $\partial/\partial\phi$ and $\partial/\partial\phi$ in $\mathscr{T}_M$ to $\partial/\partial r+\gamma_\rho\varepsilon$ and $\partial/\partial\theta+\gamma_\phi\varepsilon$ in $\mathscr{T}_P$, respectively. As γ_ρ and γ_ϕ are arbitrary, these lifts are not unique. If we require these lifts to be in $\mathscr{H}$, we obtain the unique horizontal lifts e_ρ and e_ϕ of $\partial/\partial\rho$ and $\partial/\partial\phi$, respectively. Fixing γ_ρ and γ_ϕ by requiring $\omega(e_\rho)=\omega(e_\phi)=0$, from equation (2.27) we have

$$e_\rho=\partial/\partial r\qquad e_\phi=\partial/\partial\theta-\left(\frac{r^2}{r^2+\beta^2}\right)\varepsilon. \tag{2.34}$$

This horizontal lift of $\mathscr{T}_M$ to $\mathscr{T}_P$ allows us to define a metric $\mathrm{d}s_M^2$ on M from the metric $\mathrm{d}s_P^2$. Using

$$\begin{aligned}&\mathrm{d}s_P^2(e_\rho,e_\rho)=1\qquad \mathrm{d}s_P^2(e_\rho,e_\phi)=0\\ &\mathrm{d}s_P^2(e_\phi,e_\phi)=\frac{\beta^2r^2}{r^2+\beta^2}\end{aligned} \tag{2.35}$$

we obtain

$$\mathrm{d}s_M^2=\mathrm{d}\rho^2+\left(\frac{\beta^2\rho^2}{\rho^2+\beta^2}\right)\mathrm{d}\phi^2. \tag{2.36}$$

Thus M is a Riemannian manifold with non-zero curvature (as can be checked) and infinite volume (since $\int\int_M(\rho^2+\beta^2)^{-1/2}\,\mathrm{d}\rho\,\mathrm{d}\phi$ is infinite). In order to complete the geometric discussion we express the lifts e_ρ and e_ϕ with respect to the section (2.29). By operating on functions on P which are restricted to this section, i.e. $f(r,\theta,z)=f(\rho,\theta+\sigma,\beta\sigma)$, we can write the following:

$$\begin{aligned}\frac{\partial}{\partial\rho}&=\frac{\partial}{\partial r}+\frac{\partial\sigma}{\partial\rho}\frac{\partial}{\partial\theta}+\beta\frac{\partial\sigma}{\partial\rho}\frac{\partial}{\partial z}\\ \frac{\partial}{\partial\phi}&=\frac{\partial}{\partial\theta}+\frac{\partial\sigma}{\partial\phi}\frac{\partial}{\partial\theta}+\beta\frac{\partial\sigma}{\partial\phi}\frac{\partial}{\partial z}.\end{aligned} \tag{2.37}$$

Using equations (2.32) (2.34) and (2.37) we can write the horizontal lifts in the form

$$e_\rho^{(\sigma)}=\frac{\partial}{\partial\rho}-\mathscr{A}_\rho^{(\sigma)}\varepsilon\qquad e_\phi^{(\sigma)}=\frac{\partial}{\partial\phi}-\mathscr{A}_\phi^{(\sigma)}\varepsilon. \tag{2.38}$$

Having discussed the underlying geometry of the configuration space, we now return to the quantum mechanical problem. We require to transform the Hamiltonian operators $\hat{H}_B$ and $\hat{H}$ so that they operate on functions which are defined only on the base M of the principal fibre bundle. In equation (2.7′) we observe that the kinetic energy terms of the integrand constitute the inverse metric on P. This is a crucial observation when we write the following identity which represents the horizontal-vertical decomposition of the inverse metric:

$$\frac{\partial f_1}{\partial r}\frac{\partial f_2}{\partial r}+\frac{1}{r^2}\frac{\partial f_1}{\partial \theta}\frac{\partial f_2}{\partial \theta}+\frac{\partial f_1}{\partial z}\frac{\partial f_2}{\partial z}=e_\rho f_1 e_\rho f_2+\left(\frac{1}{\rho^2}+\frac{1}{\beta^2}\right)e_\phi f_1 e_\phi f_2+\left(\frac{1}{\rho^2+\beta^2}\right)\varepsilon f_1 \varepsilon f_2. \tag{2.39}$$

Here f_1 and f_2 are any functions of r, θ and z and $\rho = r$, $\phi = \theta - z/\beta$. (The decomposition (2.39) is the same as used in the dimensional reduction of Kaluza-Klein theories [9].)

In the boson-only case the constraint (2.8′) on the complex valued wavefunctions ψ_1, ψ_2 can be written as

$$\varepsilon\psi_i = 0. \tag{2.40}$$

This constraint is satisfied when ψ_1 and ψ_2 are constant on fibres of P, i.e. $\psi_i = \psi_i(\rho, \phi) = \psi_i(r, \theta - z/\beta)$. Operating on such ψ_i, the horizontal lifts take the form

$$e_\rho\psi_i = \partial\psi_i/\partial\rho \qquad e_\phi\psi_i = \partial\psi_i/\partial\phi. \tag{2.41}$$

Using equations (2.39) (2.40) and (2.41) in equation (2.7′), integrating over the fibres, as the integrand is constant on fibres, and changing to the variables ρ, ϕ, then $\hat{H}_B$ is given by

$$(\psi_1, \hat{H}_B\psi_2) = \int_0^\infty d\rho \int_0^{2\pi} d\phi\, 2\pi\beta\rho \times\left[\frac{1}{2}\left(\frac{\partial\psi_1^*}{\partial\rho}\frac{\partial\psi_2}{\partial\rho}+\left(\frac{1}{\rho^2}+\frac{1}{\beta^2}\right)\frac{\partial\psi_1^*}{\partial\phi}\frac{\partial\psi_2}{\partial\phi}\right)+V(\rho,\phi)\psi_1^*\psi_2\right]. \tag{2.42}$$

The invariant volume element for ds_M^2 is $\beta\rho(\rho^2+\beta^2)^{-1/2}\,d\rho\,d\phi$. To obtain this we carry out the transformation

$$\psi_i = [4\pi^2(\rho^2+\beta^2)]^{-1/4}\tilde{\psi}_i. \tag{2.43}$$

The transformed Hamiltonian can be written in the form

$$(\tilde{\psi}_1, \tilde{H}_B\tilde{\psi}_2) = (\psi_1, \hat{H}_B\psi_2) = \int_0^\infty d\rho\int_0^{2\pi} d\phi \frac{\beta\rho}{\sqrt{\rho^2+\beta^2}} \times\left\{\frac{1}{2}\left[\frac{\partial\tilde{\psi}_1^*}{\partial\rho}\frac{\partial\tilde{\psi}_2}{\partial\rho}+\left(\frac{1}{\rho^2}+\frac{1}{\beta^2}\right)\frac{\partial\tilde{\psi}_1^*}{\partial\phi}\frac{\partial\tilde{\psi}_2}{\partial\phi}\right]+\tilde{V}\tilde{\psi}_1^*\tilde{\psi}_2\right\} \tag{2.44}$$

or alternatively

$$\tilde{H}_B = -\tfrac{1}{2}\Delta_M + \tilde{V} \tag{2.45}$$

where Δ_M is the Laplacian on M for the metric ds_M^2 and $\tilde{V}$ is given by

$$\tilde{V} = V + \frac{4\beta^2-\rho^2}{8(\rho^2+\beta^2)^2}. \tag{2.46}$$

Thus by writing the boson Hamiltonian in terms of the natural geometry of the base M we obtain an extra operator ordering term to the potential.

For the boson–fermion case the constraint (2.20) on the $\mathbb{C}^4$-valued wavefunctions ψ_1 and ψ_2 can be written as

$$\varepsilon\psi_i = \mathrm{i}/2\begin{pmatrix}\boldsymbol{\sigma}_3 & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{pmatrix}\psi_i. \tag{2.47}$$

This may be written in the equivalent form

$$\frac{\partial\psi_i}{\partial\alpha}(r, \theta+\alpha, z+\beta\alpha) = \mathrm{i}/2\begin{pmatrix}\boldsymbol{\sigma}_3 & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{pmatrix}\psi_i(r, \theta+\alpha, z+\beta\alpha). \tag{2.48}$$

Hence, the complete solution to the constraint, defining the wavefunctions along the fibres of P, is

$$\psi_i(r, \theta+\alpha, z+\beta\alpha) = \begin{pmatrix}\begin{matrix}\mathrm{e}^{\mathrm{i}\alpha} & 0\\ 0 & \mathrm{e}^{-\mathrm{i}\alpha}\end{matrix} & \mathbf{0}\\ \mathbf{0} & \mathbb{1}\end{pmatrix}\psi_i(r, \theta, z). \tag{2.49}$$

where $\mathbb{1}$ is the 2×2 identity matrix. We write the boson–fermion Hamiltonian in the form

$$(\psi_1, \hat{H}\psi_2) = \int_0^\infty r\,\mathrm{d}r \int_0^{2\pi}\mathrm{d}\theta \int_0^{2\pi\beta}\mathrm{d}z\mathscr{I} \tag{2.50}$$

where the integrand is

$$\mathscr{I} = \frac{1}{2}\left(\frac{\partial\psi_1^\dagger}{\partial r}\frac{\partial\psi_2}{\partial r} + \frac{1}{r^2}\frac{\partial\psi_1^\dagger}{\partial\theta}\frac{\partial\psi_2}{\partial\theta} + \frac{\partial\psi_1^\dagger}{\partial z}\frac{\partial\psi_2}{\partial z}\right) + V\psi_1^\dagger\psi_2 + \psi_1^\dagger\hat{H}_F\psi_2. \tag{2.51}$$

Then the constraint (2.49) implies that $\mathscr{I}$ is constant on fibres, i.e. $\mathscr{I} = \mathscr{I}(\rho, \phi) = \mathscr{I}(r, \theta - z/\beta)$. Changing variables to ρ, ϕ in equation (2.50) and integrating over the fibres we have

$$(\psi_1, \hat{H}\psi_2) = \int_0^\infty \mathrm{d}\rho \int_0^{2\pi}\mathrm{d}\phi\; 2\pi\beta\rho\mathscr{I}(\rho, \phi). \tag{2.52}$$

Even though $\mathscr{I}$, subject to equation (2.49), is constant on fibres of P, terms which constitute $\mathscr{I}$, i.e. ψ_1, ψ_2 and H_F, are not constant on fibres. Thus in order to write out $\mathscr{I}(\rho, \phi)$ of equation (2.52) in full we must use a section of the principal fibre bundle. We choose to use the family of sections given by equation (2.29). Thus using equations (2.29) (2.39) and (2.51), $\mathscr{I}(\rho, \phi)$, in equation (2.52), takes the form

$$\begin{aligned}\mathscr{I}(\rho, \phi) = \frac{1}{2}\Bigg[& e_\rho^{(\sigma)}\psi_1^{(\sigma)\dagger}e_\rho^{(\sigma)}\psi_2^{(\sigma)} + \left(\frac{1}{\rho^2}+\frac{1}{\beta^2}\right)e_\phi^{(\sigma)}\psi_1^{(\sigma)\dagger}e_\phi^{(\sigma)}\psi_2^{(\sigma)}\\ & + \left(\frac{1}{\rho^2+\beta^2}\right)\varepsilon\psi_1^{(\sigma)\dagger}\varepsilon\psi_2^{(\sigma)}\Bigg] + V(\rho, \phi)\psi_1^{(\sigma)\dagger}\psi_2^{(\sigma)} + \psi_1^{(\sigma)\dagger}\hat{H}_F^{(\sigma)}\psi_2^{(\sigma)}\end{aligned} \tag{2.53}$$

where

$$\psi_i^{(\sigma)}(\rho, \phi) = \psi_i(\rho, \phi + \sigma(\rho, \phi), \beta\sigma(\rho, \phi)) \tag{2.54}$$

$$\hat{H}_F^{(\sigma)} = -\frac{\mu\rho}{2}\begin{pmatrix}\begin{matrix}0 & \mathrm{e}^{\mathrm{i}(\phi+\sigma)}\\ \bar{\mathrm{e}}^{\mathrm{i}(\phi+\sigma)} & 0\end{matrix} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{pmatrix}. \tag{2.55}$$

Again, the volume element in equation (2.52) is replaced by the invariant volume element on M associated with $\mathrm{d}s^2_M$ by carrying out the transformation (2.43). Also using equation (2.38) for $e^{(\sigma)}_\rho\cdot$, $e^{(\sigma)}_\phi\cdot$ and the constraint (2.47) for $\varepsilon\cdot$ in equations (2.52) and (2.53) the Hamiltonian can be written in the form

$$(\psi_1, \hat{H}\psi_2) = (\tilde{\psi}^{(\sigma)}_1, \tilde{H}^{(\sigma)}\tilde{\psi}^{(\sigma)}_2)$$
$$= \int_0^\infty \mathrm{d}\rho \int_0^{2\pi} \mathrm{d}\phi \frac{\beta\rho}{\sqrt{\rho^2+\beta^2}}$$
$$\times\left\{\frac{1}{2}\left[\left(\left(\frac{\partial}{\partial\rho}-\frac{\mathrm{i}}{2}\begin{pmatrix}\boldsymbol{\sigma}_3 & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{pmatrix}\mathscr{A}^{(\sigma)}_\rho\right)\tilde{\psi}^{(\sigma)}_1\right)^\dagger\left(\frac{\partial}{\partial\rho}-\frac{\mathrm{i}}{2}\begin{pmatrix}\boldsymbol{\sigma}_3 & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{pmatrix}\mathscr{A}^{(\sigma)}_\rho\right)\tilde{\psi}^{(\sigma)}_2\right.\right.$$
$$\left.+\left(\frac{1}{\rho^2}+\frac{1}{\beta^2}\right)\left(\left(\frac{\partial}{\partial\phi}-\frac{\mathrm{i}}{2}\begin{pmatrix}\boldsymbol{\sigma}_3 & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{pmatrix}\mathscr{A}^{(\sigma)}_\phi\right)\tilde{\psi}^{(\sigma)}_1\right)^\dagger\left(\frac{\partial}{\partial\phi}-\frac{\mathrm{i}}{2}\begin{pmatrix}\boldsymbol{\sigma}_3 & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{pmatrix}\mathscr{A}^{(\sigma)}_\phi\right)\tilde{\psi}^{(\sigma)}_2\right]$$
$$\left.+\tilde{V}\tilde{\psi}^{(\sigma)\dagger}_1\tilde{\psi}^{(\sigma)}_2+\tilde{\psi}^{(\sigma)\dagger}_1\tilde{H}^{(\sigma)}_F\tilde{\psi}^{(\sigma)}_2\right\}. \tag{2.56}$$

Here $\mathscr{A}^{(\sigma)}_\rho$, $\mathscr{A}^{(\sigma)}_\phi$ are given by equation (2.32), $\tilde{V}$ is given by equation (2.46) and, with equation (2.55), $\tilde{H}^{(\sigma)}_F$ is given by

$$\tilde{H}^{(\sigma)}_F = \hat{H}^{(\sigma)}_F + \frac{1}{8(\rho^2+\beta^2)}\begin{pmatrix}\mathbb{1} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{pmatrix}. \tag{2.57}$$

For each σ, the Hamiltonian $\tilde{H}^{(\sigma)}$ gives the quantum mechanics of the boson–fermion model and is defined entirely on the base M of the principal fibre bundle. We make the following observations. First, the scalar potential V has become $\tilde{V}$, the same expression as for the boson-only model. Secondly, the fermion potential $\tilde{H}^{(\sigma)}_F$ contains an additional operator ordering term which we can interpret as the interaction of four fermions as the matrix in the second term in equation (2.57) can be written in the form $(\hat{\xi}^\dagger\boldsymbol{\sigma}_3\hat{\xi})(\hat{\xi}^\dagger\boldsymbol{\sigma}_3\hat{\xi})$. Thirdly, the components of the U(1) bundle connection, $\mathscr{A}^{(\sigma)}_\rho$, $\mathscr{A}^{(\sigma)}_\phi$ appear in $\tilde{H}^{(\sigma)}$ due to the fermionic part of the constraint. We could interpret the terms of $\tilde{H}^{(\sigma)}$ in which the connection components appear as resulting from the boson particle moving in M under the 'electromagnetic' field given by equation (2.33) but where the 'charge coupling' is fermionic. Fourthly, even though we have overall section independence, i.e. $(\tilde{\psi}^{(\sigma)}_1, \tilde{H}^{(\sigma)}\tilde{\psi}^{(\sigma)}_2)$ is independent of σ, $\tilde{H}^{(\sigma)}$ does depend on σ and there is no preferred choice for σ. The section σ plays the same role here as gauge fixing does in gauge field theory. Two convenient choices for the section are:

(*a*) the $z=0$ plane: $r=\rho$, $\theta=\phi$, $z=0$ (i.e. $\sigma=0$);

(*b*) the $y=0$ plane: $r=\rho$, $\theta=0$, $z=-\beta\phi$ (i.e. $\sigma=-\phi$).

($\tilde{H}^{(\sigma=0)}$ and $\tilde{H}^{(\sigma=-\phi)}$ are not equal, as can be checked.)

3. A general class of finite-dimensional gauge invariant models

In this section we consider the generalisation of the models of the previous section, i.e. where the configuration space becomes a general principal fibre bundle. This is the case B referred to in the introduction. By taking this general viewpoint we will be able to identify the geometric nature of the operator ordering terms arising (which was not possible for the models of the previous section due to their specific nature). First, we discuss relevant geometric features of the configuration space and then consider the generalisation of the Hamiltonian operators and constraint operators for the boson and boson-fermion models.

We take the configuration space to be any connected manifold P on which a compact Lie group G acts freely from the right so as to give the structure of a principal fibre bundle [9, 13]. The fibres of P are the orbits of G and by identifying points on fibres of P we obtain the base M of the principal fibre bundle. As we are interested only in local expressions for operator ordering terms we will work with local coordinates henceforth. Let $\boldsymbol{\alpha}$, $\boldsymbol{x}$ and $\boldsymbol{u}$ denote local coordinates of points in G, M and P, respectively. Moreover, as P is locally a product of M and G, we can change coordinates so that a point $\boldsymbol{u}$ on the fibre in P above $\boldsymbol{x}$ in M can be transformed to take the form $(\boldsymbol{x}, \boldsymbol{\alpha})$. P is also equipped with a Riemannian metric $\mathrm{d}s_P^2 = k = k_{AB}(\boldsymbol{u})\,\mathrm{d}u^A\,\mathrm{d}u^B$, which is G-invariant.

The Lie algebra of G, denoted $\mathscr{G}$, we take to be spanned by $\{T_a\}$ satisfying $[T_a, T_b] = C^c_{ab} T_c$ where C^c_{ab} are the structure constants of $\mathscr{G}$. In $\mathscr{T}_P$ the tangent space to P, there is a vertical subspace parallel to the fibres, denoted $\mathscr{V}$, and spanned by $\{\varepsilon_a\}$ where, for any function on P, ε_a is given by

$$\varepsilon_a f(\boldsymbol{u}) = \frac{\mathrm{d}}{\mathrm{d}t} f(\boldsymbol{u}\cdot\exp(tT_a))|_{t=0}. \tag{3.1}$$

Because the G action on P is free, $\{\varepsilon_a\}$ is linearly independent. Moreover, we also have

$$[\varepsilon_a, \varepsilon_b] = C^c_{ab}\varepsilon_c \tag{3.2}$$

from the properties of $\{T_a\}$. Using $\{\varepsilon_a\}$ and $\mathrm{d}s_P^2$ we can define a metric on any fibre of P. Letting $h_{ab} = k(\varepsilon_a, \varepsilon_b)$, then on the fibre of P over $\boldsymbol{x}$ in M

$$\mathrm{d}s_F^2 = h_{ab}(\boldsymbol{x}, \boldsymbol{\alpha})\,\mathrm{d}\alpha^a\,\mathrm{d}\alpha^b. \tag{3.3}$$

Using $\mathrm{d}s_P^2$ we can define the horizontal subspace $\mathscr{H}$ of $\mathscr{T}_P$ as the orthogonal complement $\mathscr{V}$. Thus we have the decomposition $\mathscr{T}_P = \mathscr{V} \oplus \mathscr{H}$. As $\mathrm{d}s_P^2$ is G-invariant, it follows that $\mathscr{H}$ defines a connection on the principal fibre bundle and with respect to this connection we can horizontally lift $\partial/\partial x^i$ to e_i in $\mathscr{H}$ where $\{\partial/\partial x^i\}$ is a basis for the tangent space to M, $\mathscr{T}_M$. We can define a metric on M, $\mathrm{d}s_M^2$, using the horizontal lifts $\{e_i\}$ by letting $\gamma_{ij} = k(e_i, e_j)$ and putting

$$\mathrm{d}s_M^2 = \gamma_{ij}(\boldsymbol{x})\,\mathrm{d}x^i\,\mathrm{d}x^j. \tag{3.4}$$

Letting $\{\partial/\partial u^A\}$ be a basis for $\mathscr{T}_P$, then at the point $\boldsymbol{u}$ in P, on the fibre above $\boldsymbol{x}$ in M, we have the decomposition of the inverse metric with respect to $\mathscr{V}$ and $\mathscr{H}$ [9]:

$$k^{AB}(\boldsymbol{u})\frac{\partial}{\partial u^A} \otimes \frac{\partial}{\partial u^B} = \gamma^{ij}(\boldsymbol{x}) e_i \otimes e_j + h^{ab}(\boldsymbol{u})\varepsilon_a \otimes \varepsilon_b \tag{3.5}$$

using $\{e_i\}$ and $\{\varepsilon_a\}$ as bases for $\mathscr{H}$ and $\mathscr{V}$. To obtain components of the connection we take local sections of the principal fibre bundle by writing $(\boldsymbol{x}, \boldsymbol{\alpha} = \sigma(\boldsymbol{x}))$ where σ is a locally defined G-valued function on M. With respect to this section the horizontal lifts are given by

$$e_i^{(\sigma)} = \frac{\partial}{\partial x^i} - \mathscr{A}_i^{(\sigma)a}(\boldsymbol{x})\varepsilon_a. \tag{3.6}$$

The three sets of functions $\{\gamma_{ij}(\boldsymbol{x})\}$, $\{\mathscr{A}_b^{(\sigma)}(\boldsymbol{x})\}$ and $\{h_{ab}^{(\sigma)}(\boldsymbol{x}) = h_{ab}(\boldsymbol{x}, \boldsymbol{\alpha} = \sigma(\boldsymbol{x}))\}$ completely and uniquely characterise all G-invariant Riemannian metrics on a principal fibre bundle as can be inferred from equations (3.5) and (3.6). In considering the quantum mechanical models below we will express the operator ordering terms with respect to these three sets of functions.

We now consider the quantum mechanics of the gauge invariant boson model with a configuration space which is the principal fibre bundle P. Because the metric on P, ds_P^2, can have non-zero curvature, an operator ordering choice needs to be made for the Hamiltonian operator for a particle moving in P in a potential V. We choose as the generalisation of the boson Hamiltonian operator (2.7) the following natural geometric one:

$$\hat{H}_{\mathrm{B}} = -\tfrac{1}{2}\Delta_k + V \tag{3.7}$$

where Δ_k is the Laplacian associated with the metric $ds_P^2 = k$. The potential V is G-invariant, i.e. $\varepsilon_a V = 0$. $\hat{H}_{\mathrm{B}}$ operates on complex-valued functions and can be written in the form

$$(\psi_1, \hat{H}_{\mathrm{B}}\psi_2) = \int_P \prod d\boldsymbol{u}\sqrt{\det k}\left(\tfrac{1}{2} k^{AB} \frac{\partial \psi_1^*}{\partial u^A} \frac{\partial \psi_2}{\partial u^B} + V\psi_1^*\psi_2\right). \tag{3.8}$$

The gauge invariance of the model manifests itself as the Gauss-law constraint on $\{\psi_i\}$ which generalises to

$$\varepsilon_a \psi_i = 0. \tag{3.9}$$

When the constraint (3.9) is applied to the integrand of equation (3.8) the term in parentheses is constant on fibres of P. Also by changing from local coordinates $\boldsymbol{u}$ to $(\boldsymbol{x}, \boldsymbol{\alpha})$, it follows from the decomposition (3.5) that

$$\int_P \prod d\boldsymbol{u}\sqrt{\det k(\boldsymbol{u})}(\,\cdot\,) = \int_M \prod d\boldsymbol{x}\sqrt{\det \gamma(\boldsymbol{x})} \int_{\mathrm{G}} \prod d\boldsymbol{\alpha}\sqrt{\det h(\boldsymbol{x}, \boldsymbol{\alpha})}(\,\cdot\,). \tag{3.10}$$

By defining the hypervolume of the fibre of P above the point $\boldsymbol{x}$ in M by

$$\nu(\boldsymbol{x}) = \int_{\mathrm{G}} \prod d\boldsymbol{\alpha}\sqrt{\det h(\boldsymbol{x}, \boldsymbol{\alpha})} \tag{3.11}$$

and using it in equation (3.8) together with the decomposition (3.5) and constraint (3.9) we have

$$(\psi_1, \hat{H}_{\mathrm{B}}\psi_2) = \int_M \prod d\boldsymbol{x}\sqrt{\det \gamma(\boldsymbol{x})}\,\nu(\boldsymbol{x})\left(\frac{1}{2}\gamma^{ij} \frac{\partial \psi_1^*}{\partial x^i} \frac{\partial \psi_2}{\partial x^j} + V\psi_1^*\psi_2\right). \tag{3.12}$$

We need to eliminate $\nu(\boldsymbol{x})$ from the integrand of equation (3.12) to obtain the correct volume element on M and we can achieve this by letting

$$\psi_i = \tilde{\psi}_i / \nu^{1/2}. \tag{3.13}$$

This transforms the Hamiltonian operator $\hat{H}_{\mathrm{B}}$ to $\tilde{H}_{\mathrm{B}}$ where

$$\begin{aligned}(\tilde{\psi}_1, \tilde{H}_{\mathrm{B}}\tilde{\psi}_2) &= (\psi_1, \hat{H}_{\mathrm{B}}\psi_2)\\ &= \int_M \prod d\boldsymbol{x}\sqrt{\det \gamma(\boldsymbol{x})}\left(\frac{1}{2}\gamma^{ij} \frac{\partial \tilde{\psi}_1^*}{\partial x^i} \frac{\partial \tilde{\psi}_2}{\partial x^j} + V\tilde{\psi}_1^*\tilde{\psi}_2\right)\end{aligned} \tag{3.14}$$

or alternatively

$$\tilde{H}_B = -\tfrac{1}{2}\Delta_\gamma + \tilde{V} \tag{3.15}$$

where Δ_γ is the Laplacian operator associated with the metric ds^2_M and

$$\tilde{V} = V + \frac{1}{4}\left(\Delta_\gamma(\ln \nu) + \frac{1}{2}\gamma^{ij}\frac{\partial \ln \nu}{\partial x^i}\frac{\partial \ln \nu}{\partial x^j}\right). \tag{3.16}$$

Equations (3.11) and (3.16) identify the operator ordering terms which have been added to the potential of the boson model with respect to $\{\gamma_{ij}\}$ and $\{h_{ab}\}$.

For the boson-fermion Hamiltonian operator we add a fermion part $\hat{H}_F$ to $\hat{H}_B$ given by equation (3.7), i.e.

$$\hat{H} = -\tfrac{1}{2}\Delta_k + V + \hat{H}_F. \tag{3.17}$$

$\hat{H}_F$, which must be gauge invariant, is constructed in the manner now described. We suppose that we have a Clifford algebra generated by $\{l_1, l_2, \ldots, l_N\}$, i.e. that

$$l_i l_j + l_j l_i = \delta_{ij} \qquad l_i^\dagger = l_i. \tag{3.18}$$

We also suppose that the Lie algebra, $\mathscr{G}$, is represented on this Clifford algebra, i.e. a generator T_a is represented by Γ_a, where (see [14])

$$\Gamma_a = \tfrac{1}{4}\mathrm{i}\Gamma_a^{ij}[l_i, l_j] \tag{3.19}$$

and Γ_a^{ij} is real and antisymmetric in i, j and such that

$$[\Gamma_a, \Gamma_b] = \mathrm{i}C_{ab}^c\Gamma_c. \tag{3.20}$$

(For the three-dimensional model of section 2 the one generator is represented by $\Gamma = \tfrac{1}{4}\mathrm{i}([l_2, l_4] - [l_1, l_3])$.) The gauge invariance of $\hat{H}_F$ is expressed by

$$[\varepsilon_a - \mathrm{i}\Gamma_a, \hat{H}_F] = 0. \tag{3.21}$$

Thus, if in generalising equation (2.15) we take

$$\hat{H}_F = \mathrm{i}w_{ij}(\boldsymbol{u})[l_i, l_j] \tag{3.22}$$

where $w_{ij}(\boldsymbol{u})$ is real valued and antisymmetric in i and j, then equation (3.21) requires that

$$\varepsilon_a w_{ij} = \Gamma_a^{jk} w_{ki} - \Gamma_a^{ik} w_{kj}. \tag{3.23}$$

(From equation (2.15) $\hat{H}_F = \tfrac{1}{4}\mathrm{i}\mu\{x([l_2, l_3] + [l_1, l_4]) + y([l_2, l_1] + [l_4, l_2])\}$ for the three-dimensional model of section 2.) $\hat{H}$ can be written in the alternative form

$$(\psi_1, \hat{H}\psi_2) = \int_P \prod \mathrm{d}\boldsymbol{u}\sqrt{\det k}\,\mathscr{I} \tag{3.24}$$

where

$$\mathscr{I} = \tfrac{1}{2}k^{AB}\frac{\partial \psi_1^\dagger}{\partial u^A}\frac{\partial \psi_2}{\partial u^B} + V\psi_1^\dagger\psi_2 + \psi_1^\dagger\hat{H}_F\psi_2 \tag{3.25}$$

and ψ_i are multicomponent complex-valued functions. The Gauss-law constraint generalising equation (2.20) is

$$(\varepsilon_a - \mathrm{i}\Gamma_a)\psi_i = 0. \tag{3.26}$$

The solution to this equation defines the behaviour of ψ_i along the fibres of P. Moreover, it then follows that $\mathscr{I}$ is constant on fibres of P. Thus we can again use equations (3.10) and (3.11) to write equation (3.24) as

$$(\psi_1, \hat{H}\psi_2) = \int_M \prod \mathrm{d}\boldsymbol{x}\sqrt{\det \gamma(\boldsymbol{x})}\,\nu(\boldsymbol{x})\mathscr{I}. \tag{3.27}$$

Because the terms which consistute $\mathscr{I}$, i.e. ψ_i and $\hat{H}_\mathrm{F}$, are not constant on fibres of P, in order to write out $\mathscr{I}$ in full we choose a local section $(\boldsymbol{x}, \boldsymbol{\alpha} = \sigma(\boldsymbol{x}))$ on P. Using this and equation (3.5) we have

$$\begin{aligned}\mathscr{I} = {} & \tfrac{1}{2}\gamma^{ij}(\boldsymbol{x})e_i^{(\sigma)}\psi_1^{(\sigma)\dagger}e_j^{(\sigma)}\psi_2^{(\sigma)} + \tfrac{1}{2}h^{(\sigma)ab}(\boldsymbol{x})\varepsilon_a\psi_1^{(\sigma)\dagger}\varepsilon_b\psi_2^{(\sigma)} \\ & + V\psi_1^{(\sigma)\dagger}\psi_2^{(\sigma)} + \psi_1^{(\sigma)\dagger}\hat{H}_\mathrm{F}^{(\sigma)}\psi_2^{(\sigma)}\end{aligned} \tag{3.28}$$

where

$$\hat{H}_\mathrm{F}^{(\sigma)} = \hat{H}_\mathrm{F}(\boldsymbol{x}, \boldsymbol{\alpha} = \sigma(\boldsymbol{x}))$$

and

$$\psi_i^{(\sigma)}(\boldsymbol{x}) = \psi_i(\boldsymbol{x}, \boldsymbol{\alpha} = \sigma(\boldsymbol{x})).$$

Using the substitution (3.13) to obtain the correct volume element for $\mathrm{d}s_M^2$ in equation (3.27) and using equation (3.6) for $e_i^{(\sigma)}\psi_{1,2}^{(\sigma)}$ and the constraint (3.26) for $\varepsilon_a\psi_{1,2}^{(\sigma)}$ in equation (3.28), the Hamiltonian becomes

$$\begin{aligned}(\psi_1, \hat{H}\psi_2) &= (\tilde{\psi}_1^{(\sigma)}, \tilde{H}^{(\sigma)}\tilde{\psi}_2^{(\sigma)}) \\ &= \int_M \prod \mathrm{d}\boldsymbol{x}\sqrt{\det \gamma}\left[\tfrac{1}{2}\gamma^{ij}\left(\left(\frac{\partial}{\partial x^i} - \mathrm{i}\mathscr{A}_i^{(\sigma)a}\Gamma_a\right)\tilde{\psi}_1^{(\sigma)}\right)^\dagger\left(\frac{\partial}{\partial x^j} - \mathrm{i}\mathscr{A}_j^{(\sigma)b}\Gamma_b\right)\tilde{\psi}_2^{(\sigma)}\right. \\ &\quad \left. + \tilde{V}\tilde{\psi}_1^{(\sigma)\dagger}\tilde{\psi}_2^{(\sigma)} + \tilde{\psi}_1^{(\sigma)\dagger}\tilde{H}_\mathrm{F}^{(\sigma)}\tilde{\psi}_2^{(\sigma)}\right].\end{aligned} \tag{3.29}$$

Here, the operator ordered potential $\tilde{V}$ is given by equation (3.16) and the operator ordered fermionic potential $\tilde{H}_\mathrm{F}^{(\sigma)}$ is given by

$$\tilde{H}_\mathrm{F}^{(\sigma)} = \hat{H}_\mathrm{F}^{(\sigma)} + \tfrac{1}{2}h^{(\sigma)ab}\Gamma_a\Gamma_b. \tag{3.30}$$

All terms in $\tilde{H}^{(\sigma)}$, as given by equations (3.16), (3.29) and (3.30) other than V and $\hat{H}_\mathrm{F}^{(\sigma)}$ are expressed with respect to the bundle connection $\mathscr{A}_i^{(\sigma)a}$, the metric on M, $\mathrm{d}s_M^2 = \gamma$, and the fibre metric, $\mathrm{d}s_\mathrm{F}^2 = h$, which uniquely characterise the G-invariant metric on P, $\mathrm{d}s_P^2 = k$.

4. Concluding remarks

We have considered classes of constrained finite-dimensional quantum models of bosons and fermions where the configuration space has the structure of a principal fibre bundle and on which there is defined a metric invariant under the group action. This geometrical structure is similar to that present in a gauge invariant field theory but without the added complications of infinite dimensionality (i.e. renormalisation and overall Lorentz covariance). For the metric on the principal fibre bundle we firstly took the Euclidean metric (case A) in section 2 (this situation is most similar to gauge field theory) and secondly we have considered the case of a general metric (case B) in section 3.

In concluding, we make the following points.

(i) The reduced boson Hamiltonian operator, $\tilde{H}_B$ (i.e. as defined on the base of the principle fibre bundle) is given for case A by equations (2.45) and (2.46) and for case B by equations (3.15) and (3.16). $\tilde{H}_B$ consists of $-\frac{1}{2}$ times the natural Laplacian on the base plus a modified potential.

(ii) The reduced boson-fermion Hamiltonian operator, $\tilde{H}^{(\sigma)}$ is given for case A by equations (2.32), (2.46), (2.55), (2.56) and (2.57) and for case B by equations (3.16), (3.29) and (3.30). $\tilde{H}^{(\sigma)}$ consists of: (*a*) $-\frac{1}{2}$ times the natural Laplacian on the base; the Laplacian now is modified by being minimally coupled to the fermion fields through the connection induced from the group action and the invariant metric; (*b*) a modified potential as for $\tilde{H}_B$; and (*c*) a fermionic potential, modified by the addition of a four-fermion interaction term. There is an added complication in writing out $\tilde{H}^{(\sigma)}$ as compared with $\tilde{H}_B$ because local sections σ are needed (even though there is overall σ-independence). The role of σ here is the same as that of gauge fixing in field theory.

(iii) We have used the vertical-horizontal decomposition of the tangent space to the principal fibre bundle (resulting from the group action and the invariant metric) to remove the redundant degrees of freedom from the constrained system and obtain a Hamiltonian operator (containing operator order terms) on the base of the principle fibre bundle. The general metric case was considered in section 3 and in equations (3.11), (3.16), (3.29) and (3.30) the operator ordering terms are expressed in terms of $\{\mathcal{A}_i^{(\sigma)a}, h_{ab}^{(\sigma)}, \gamma_{ij}\}$. This set of functions uniquely characterises the original group invariant metric k on the principal fibre bundle (see [9] and references therein). This characterisation is the same as occurs in the dimensional reduction of Kaluza-Klein theories (there $\{\mathcal{A}_i^{(\sigma)a}\}$ would be the set of gauge potentials, $\{\gamma_{ij}\}$ would define the metric on spacetime and $\{h_{ab}^{(\sigma)}\}$ would constitute the set of Brans-Dicke scalar fields [9, 11]. Furthermore, the characterisation in terms of $\{\mathcal{A}_i^{(\sigma)a}, h_{ab}^{(\sigma)}, \gamma_{ij}\}$ obviates the need to express the operator ordering terms in other ways (i.e. in terms of the curvatures of the principal fibre bundle or the base or the geometry of the embedding of the orbits, etc, and which was the approach taken in [8]).

The gauge invariant quantum models that we have dealt with in this paper have been finite dimensional. This was done in order to be able to present the geometry of the operator ordering terms in a clear way without the additional difficulties of regularising the divergences of a field theory. However, it is in the infinite-dimensional case, i.e. a field theory, where gauge invariance occurs in practice. Thus, we address our final remarks on how the results of the paper can be extended for gauge invariant field theory. For this case, the expressions derived above for the operator ordering terms are true only in a formal sense due to the presence of many divergent factors. (A full discussion of the divergences that arise in a gauge invariant field theory from the geometric point of view is given in [3].) In order to resolve the divergence problem, two methods are available. The first method is to consider lattice gauge theory and take the continuum limit, as in [8]. The second method consists of the introduction of a space-volume cut-off and a gauge covariant continuum ultraviolet cut-off and to construct the Feynman-Kac integral for a gauge invariant field theory through a suitable Brownian motion on the space of gauge group orbits [3, 15].

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